Pinching, Pontrjagin classes, and negatively curved vector bundles

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Abstract

We prove several finiteness results for the class $\mathcal{M}_{a,b,\pi,n}$ of n-manifolds that have fundamental groups isomorphic to π and that can be given complete Riemannian metrics of sectional curvatures within $[a,b]$ where $a \le b < 0$. In particular, if M is a closed negatively curved manifold of dimension at least three, then only finitely many manifolds in the class $\mathcal{M}_{a,b,\pi_1(M),n}$ are total spaces of vector bundles over M. Furthermore, given a word-hyperbolic group π and an integer n there exists a positive $\epsilon = \epsilon(n,\pi)$ such that the tangent bundle of any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ has zero rational Pontrjagin classes.

1 Introduction

According to the Cartan-Hadamard theorem, the universal cover of any complete negatively curved manifold is diffeomorphic to the Euclidean space. Surprisingly, beyond this fact little is known about topology of *infinite* volume complete negatively curved manifolds. For example, so far there has been found no restriction on the fundamental groups of such manifolds except being the fundamental groups of aspherical manifolds.

In this paper we study topology of pinched negatively curved manifolds. (We call a manifold *pinched negatively curved* if it admits a complete Riemannian metric with sectional curvatures bounded between two negative constants.) For instance, the fundamental groups of pinched negatively curved manifolds have the property that any amenable subgroup must be finitely generated and virtually nilpotent [BS87, Bow93]. Examples include closed negatively curved manifolds and complete locally symmetric negatively curved manifolds. Also, elementary warped product construction gives pinched negatively curved metrics on the direct products of pinched negatively curved manifolds with Euclidean spaces [FJ87]. Furthermore, the total space of any vector bundle over a closed negatively curved manifold is pinched negatively curved [And87].

According to a general pinching principle, a negatively curved manifold with sectional curvatures close to -1 ought to be topologically similar to a hyperbolic manifold. In its strongest form this principle fails even for closed manifolds. In fact, for each $n \geq 4$ and any $\epsilon > 0$, M. Gromov and W. Thurston [GT87] found a closed negatively curved n-manifold with sectional curvatures within $[-1-\epsilon, -1]$ that is not diffeomorphic to a hyperbolic manifold. (Any closed negatively curved 3-manifold is diffeomorphic to a hyperbolic one if Thurston's hyperbolization conjecture is true.) However, the pinching principle holds if the pinching constant ϵ is allowed to depend on the fundamental group, namely, there exists an $\epsilon = \epsilon(\pi) > 0$ such that any closed manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ is diffeomorphic to a hyperbolic manifold [Bel97b]. In even dimensions Gromov [Gro78] proved a stronger pinching theorem: any closed even-dimensional Riemannian manifold with sectional curvatures within $[-1-\epsilon, -1]$ is diffeomorphic to a hyperbolic manifold where ϵ depends on the dimension and the Euler characteristic of the manifold.

The following theorem restricts the topology of strongly pinched negatively curved manifolds without assuming compactness.

Theorem 1.1. *Let π be the fundamental group of a finite aspherical cell complex. Suppose that π is not virtually nilpotent and that π does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent group. Then, for any positive integer n, there exists an $\epsilon = \epsilon(\pi, n) > 0$ such that any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ is tangentially homotopy equivalent to an n-manifold of constant negative curvature.*

Recall that a homotopy equivalence of manifolds $f : M \to N$ is called tangential if the vector bundles $f^{\#}TN$ and TM are stably isomorphic. If we weaken the assumption "π is the fundamental group of a finite aspherical cell complex" to "π is finitely presented", then the same conclusion holds without the word "tangentially".

Any manifold of constant sectional curvature has zero rational Pontrjagin classes [Ave70], hence, applying the Mayer-Vietoris sequence and the accessibility result of Delzant and Potyagailo [DP98] (reviewed in 2.5), we deduce the following.

Theorem 1.2. *Let π be a finitely presented group with finite $4k$th Betti numbers for all $k > 0$. Assume that any nilpotent subgroup of π has cohomological dimension ≤ 2. Then for any n there exist $\epsilon = \epsilon(n, \pi) > 0$ such that the tangent bundle of any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ has zero rational Pontrjagin classes.*

In particular, the following is true because any nilpotent subgroup of a word-hyperbolic group has cohomological dimension ≤ 1.

Corollary 1.3. *Let π be a word-hyperbolic group. Then for any n there exist $\epsilon = \epsilon(n, \pi) > 0$ such that the tangent bundle of any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ has zero rational Pontrjagin classes.*

As we showed in [Bel97a], the class $\mathcal{M}_{a,b,\pi,n}$ falls into finitely many tangentially homotopy types under mild assumptions on the group π. However, [Bel97a] does not provide explicit bounds on the number of tangentially homotopy inequivalent manifolds in the class $\mathcal{M}_{a,b,\pi,n}$. The only exception is the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ in the theorem 1.1 because the number of tangentially homotopy inequivalent real hyperbolic manifolds is bounded by the number of connected components of the representation variety $\mathrm{Hom}(\pi, \mathrm{Isom}(\mathbf{H}^n_{\mathbb{R}}))$ [Bel98]. The latter can be estimated in terms of n and the numbers of generators and relators of π.

M. Anderson [And87] showed that the total space of any vector bundle over a closed negatively curved manifold M can be given a complete Riemannian metric of sectional curvature within $[a, b]$ for some $a \le b < 0$. However, the theorem below says that, once pinching a/b is fixed, total spaces of only finitely many vector bundles over M of a given rank admit metrics of sectional curvature within $[a, b]$. Note that the set of isomorphism classes of vector bundles of the same rank over M is *infinite* provided certain Betti numbers of M are nonzero (see 4.15).

Theorem 1.4. *Let M be a closed negatively curved manifold with* $\dim(M) \ge 3$ *and let $k > 0$ be a positive integer and $a \le b < 0$ be real numbers. Then, up to isomorphism, there exist only finitely many rank k vector bundles over M whose total spaces admit complete Riemannian metrics with sectional curvatures within $[a, b]$.*

As we explain in the appendix, the isomorphism type of a vector bundle ξ over a finite cell complex is determined, up to finitely many possibilities, by its Euler class and Pontrjagin classes (for non-orientable bundles one looks at the Euler class of the orientable two-fold-pullback). Pontrjagin classes depend on the tangent bundle of the total space $E(\xi)$ while the Euler class can be computed via intersections in $E(\xi)$. Finiteness results for tangent bundles of pinched negatively curved manifolds were obtained in [Bel97a]; in the present paper we prove similar results for intersections.

A homotopy equivalence $f : N \to L$ of orientable n-manifolds is called *intersection preserving* if, for some orientations of N and L, the intersection number of any pair of homology classes of N is equal to the intersection number of their f-images in L. More generally, a homotopy equivalence f of nonorientable

manifolds is called *intersection preserving* if f preserves first Stiefel-Whitney classes and the lift of f to orientable two-fold covers is an intersection preserving homotopy equivalence. For example, if f is homotopic to a homeomorphism, then f is an intersection preserving homotopy equivalence.

Theorem 1.5. *Let π be a finitely presented group with finite Betti numbers. Assume that π does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent group. Then, for any $a \le b < 0$, the class $\mathcal{M}_{a,b,\pi,n}$ breaks into finitely many intersection preserving homotopy types.*

Theorem 1.5 was first proved in [Bel98] for oriented locally symmetric negatively curved manifolds. This special case is somewhat easier to handle due to the fact that theory of convergence discussed in the section 2 has been thoroughly studied in the constant negative curvature case. Note that if $a = b = -1$, one can naturally identify $\mathcal{M}_{a,b,\pi,n}$ with the set of injective discrete representations of π into the isometry group of the real hyperbolic n-space which is a standard object in Kleinian group theory.

Synopsis of the paper. The second section contains background in convergence of Riemannian manifolds as well as some results on splittings and accessibility over virtually nilpotent groups. The main technical result is proved in the section three after discussing some invariants of maps and proper discontinuous actions. The forth section is devoted to applications. Section five is a discussion of certain natural pinching invariants. A bundle-theoretic result is proved in the appendix.

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2 Convergence, splittings and accessibility

The exposition in this section is a variation of the one given in [Bel97a]. Several straightforward lemmas are only stated and refered to [Bel97a] for proofs.

By an *action* of an abstract group π on a space X we mean a group homomorphism $\rho : \pi \to \mathrm{Homeo}(X)$. An action ρ is called *free* if $\rho(\gamma)(x) \neq x$ for all $x \in X$ and all $\gamma \in \pi \setminus \mathrm{id}$. In particular, if ρ is a free action, then ρ is injective.

2.1 Equivariant pointed Lipschitz topology

Let Γ_k be a discrete subgroup of the isometry group of a complete Riemannian manifold X_k and p_k be a point of X_k. The class of all such triples $\{(X_k, p_k, \Gamma_k)\}$ can be given the so-called equivariant pointed Lipschitz topology [Fuk86]; when Γ_k is trivial this reduces to the usual pointed Lipschitz topology. For convenience of the reader we give here some definitions borrowed from [Fuk86].

For a group Γ acting on a pointed metric space (X, p, d) the set $\{\gamma \in \Gamma : d(p, \gamma(p)) < r\}$ is denoted by $\Gamma(r)$. An open ball in X of radius r with center at p is denoted by $B_r(p, X)$.

For $i = 1, 2$, let (X_i, p_i) be a pointed complete metric space with the distance function d_i and let Γ_i be a discrete group of isometries of X_i. In addition, assume that X_i is a C^∞–manifold. Take any $\epsilon > 0$.

Then a quadruple $(f_1, f_2, \phi_1, \phi_2)$ of maps $f_i : B_{1/\epsilon}(p_i, X_i) \to B_{1/\epsilon}(p_{3-i}, X_{3-i})$ and $\phi_i : \Gamma_i(1/3\epsilon) \to \Gamma_{3-i}$ is called an *ϵ–Lipschitz approximation* between the triples (X_1, p_1, Γ_1) and (X_2, p_2, Γ_2) if the following seven condition hold:

- f_i is a diffeomorphism onto its image;
- for each $x_i \in B_{1/3\epsilon}(p_i, X_i)$ and every $\gamma_i \in \Gamma_i(1/3\epsilon)$, $f_i(\gamma_i(x_i)) = \phi_i(\gamma_i)(f_i(x_i))$;
- for every $x_i, x_i' \in B_{1/\epsilon}(p_i, X_i)$, $e^{-\epsilon} < d_{3-i}(f_i(x_i), f_i(x_i'))/d_i(x_i, x_i') < e^{\epsilon}$;
- $f_i(B_{1/\epsilon}(p_i, X_i)) \supset B_{(1/\epsilon)-\epsilon}(p_{3-i}, X_{3-i})$ and $\phi_i(\Gamma_i(1/3\epsilon)) \supset \Gamma_{3-i}(1/3\epsilon - \epsilon)$;
- $f_i(B_{(1/\epsilon)-\epsilon}(p_i, X_i)) \supset B_{1/\epsilon}(p_{3-i}, X_{3-i})$ and $\phi_i(\Gamma_i(1/3\epsilon - \epsilon)) \supset \Gamma_{3-i}(1/3\epsilon)$;
- $f_{3-i} \circ f_i|_{B_{(1/\epsilon)-\epsilon}(p_i, X_i)} = \mathrm{id}$ and $\phi_{3-i} \circ \phi_i|_{\Gamma_i(1/3\epsilon-\epsilon)} = \mathrm{id}$;
- $d_{3-i}(f_i(p_i), p_{3-i}) < \epsilon$.

We say a sequence of triples (X_k, p_k, Γ_k) *converges* to (X, p, Γ) in the equivariant pointed Lipschitz topology if for any $\epsilon > 0$ there is $k(\epsilon)$ such that for all $k > k(\epsilon)$, there exists an ϵ–Lipschitz approximation between (X_k, p_k, Γ_k) and (X, p, Γ). If all the groups Γ_k are trivial, then Γ is trivial; in this case we say that that (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology.

Note that if X_k is a complete Riemannian manifold for all k, then the space X is necessarily a C^∞–manifold with a complete $C^{1,\alpha}$–Riemannian metric [GW88]. If each X_k is a Hadamard manifold, then for any $x_k \in X_k$, the sequence

(X_k, x_k) is precompact in the pointed Lipschitz topology because the injectivity radius of X_k at x_k is uniformly bounded away from zero [Fuk86, p.132].

Remark 2.1. Equivariant pointed Lipschitz topology is closely related to the so-called Chabauty topology used in Kleinian group theory [CEG84], [BP92]. Indeed, let Γ_k be a sequence of discrete subgroups of the isometry group of a complete Riemannian manifold X (e.g. a hyperbolic space). Then Γ_k converges in the Chabauty topology to a *discrete* group Γ if and only if for each $p \in X$ (X, Γ_k, p) converges to (X, Γ, p) in the equivariant pointed Lipschitz topology.

2.2 Pointwise convergence topology

Suppose that, for some $p_k \in X_k$, the sequence (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology, i.e., for any $\epsilon > 0$ there is $k(\epsilon)$ such that for all $k > k(\epsilon)$, there exists an ϵ–Lipschitz approximation (f_k, g_k) between (X_k, p_k) and (X, p). We say that a sequence $x_k \in X_k$ *converges* to $x \in X$ if for some ϵ

$$d(f_k(x_k), x) \to 0 \quad \text{as} \quad k \to \infty$$

where $d(\cdot, \cdot)$ is the distance function on X and f_k comes from the ϵ–Lipschitz approximation (f_k, g_k) between (X_k, p_k) and (X, p). Trivial examples: if (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology, then p_k converges to p; furthermore, if $x \in X$, the sequence $g_k(x)$ converges to x.

Given a sequence of isometries $\gamma_k \in \mathrm{Isom}(X_k)$ we say that γ_k *converges*, if for any $x \in X$ and any sequence $x_k \in X_k$ that converges to x, $\gamma_k(x_k)$ converges. The limiting transformation γ that takes x to the limit of $\gamma_k(x_k)$ is necessarily an isometry of X. Furthermore, if γ_k and γ'_k converge to γ and γ' respectively, then $\gamma_k \cdot \gamma'_k$ converges to $\gamma \cdot \gamma'$. In particular, γ_k^{-1} converges to γ^{-1} since the identity maps $\mathrm{id}_k : X_k \to X_k$ converge to $\mathrm{id} : X \to X$.

Let $\rho_k : \pi \to \mathrm{Isom}(X_k)$ be a sequence of isometric actions of a group π on X_k. We say that a sequence of actions (X_k, p_k, ρ_k) *converges in the pointwise convergence topology* if $\rho_k(\gamma)$ converges for every $\gamma \in \pi$. The limiting map $\rho : \Gamma \to \mathrm{Isom}(X)$ that takes γ to the limit of $\rho_k(\gamma)$ is necessarily a homomorphism. If π is generated by a finite set S, then in order to prove that ρ_k converges in the pointwise convergence topology it suffices to check that $\rho_k(\gamma)$ converges, for every $\gamma \in S$.

It is worth clarifying that the term "pointwise convergence" refers to the convergence of group action rather than individual isometries. The definitions are set up so that individual isometries converge "uniformly on compact subsets". The motivation comes from the following example.

Example 2.2. Let X be a complete Riemannian manifold (e.g. a hyperbolic space). Consider the isometry group $\mathrm{Isom}(X)$ equipped with compact-open topology and let π be a group. The space $\mathrm{Hom}(\pi, \mathrm{Isom}(X))$ has a natural topology (which is usually called "algebraic topology" or "pointwise convergence topology"), namely ρ_k is said to converge to ρ if, for each $\gamma \in \pi$, $\rho_k(\gamma)$ converges to $\rho(\gamma)$ in the Lie group $\mathrm{Isom}(X)$. Note that if π is finitely generated, this topology on $\mathrm{Hom}(\pi, \mathrm{Isom}(X))$ coincide with the compact-open topology. Certainly, for any $p \in X$, the constant sequence (X, p) converges to itself in pointed Lipschitz topology. Then the sequence (X, p, ρ_k) converges in the pointwise convergence topology (as defined in this section) if and only if $\rho_k \in \mathrm{Hom}(\pi, \mathrm{Isom}(X))$ converges in the algebraic topology. (Indeed, $\rho_k(\gamma)$ converges to $\rho(\gamma)$ in $\mathrm{Isom}(X)$ iff $\rho_k(\gamma)$ converges to $\rho(\gamma)$ uniformly on compact subsets. In particular, the latter implies that $\rho_k(\gamma)(x_k) \to \rho(\gamma)(x)$ for any $x_k \to x$. Conversely, if $\rho_k(\gamma)(x_k) \to \rho(\gamma)(x)$ for any $x_k \to x$, then $\rho_k(\gamma)$ converges to $\rho(\gamma)$ uniformly on compact subsets [KN63, 4.7, Lemma 5].)

A sequence of actions (X_k, p_k, ρ_k) is called *precompact in the pointwise convergence topology* if every subsequence of (X_k, p_k, ρ_k) has a subsequence that converges in the pointwise convergence topology.

Repeating the proof of [KN63, 4.7], it is easy to check that a sequence of isometries $\gamma_k \in \mathrm{Isom}(X_k)$ has a converging subsequence if, for some converging sequence $x_k \in X_k$, the sequence $d_k(x_k, \gamma_k(x_k))$ is bounded (where $d_k(\cdot, \cdot)$ is the distance function on X_k).

Suppose that π is a countable (e.g. finitely generated) group and assume that for each $\gamma \in \pi$ the sequence $d_k(p_k, \rho_k(\gamma)(p_k))$ is bounded. Then (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology. (Indeed, let $\gamma_1 \dots \gamma_n \dots$ be the list of all elements of π. Take any subsequence $\rho_{k,0}$ of ρ_k. Pass to subsequence $\rho_{k,1}$ of $\rho_{k,0}$ so that $\rho_{k,1}(\gamma_1)$ converges. Then pass to subsequence $\rho_{k,2}$ of $\rho_{k,1}$ such that $\rho_{k,2}(\gamma_2)$ converges, etc. Then $\rho_{k,k}(\gamma_n)$ converges for every n.)

Note that if π is generated by a finite set S, then to prove that ρ_k is precompact it suffices to check that $d_k(p_k, \rho_k(\gamma)(p_k))$ is bounded, for all $\gamma \in S$ because it implies that $d_k(p_k, \rho_k(\gamma)(p_k))$ is bounded, for each $\gamma \in \pi$.

The following two lemmas can be easily deduced from definitions; the reader is refered to [Bel97a] for details.

Lemma 2.3. *Let $\rho_k : \pi \to \mathrm{Isom}(X_k)$ be a sequence of isometric actions of a discrete group π on complete Riemannian n-manifolds X_k such that $\rho_k(\pi)$ acts freely. If the sequence $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed*

Lipschitz topology to (X,Γ,p) and (X_k,p_k,ρ_k) converges to (X,p,ρ) in the pointwise convergence topology, then

(1) Γ *acts freely, and*

(2) $\rho(\pi)\subset\Gamma$, *and*

(3) $\ker(\rho)\subset\ker(\rho_k)$, *for all large* k.

Lemma 2.4. *Let $\rho_k:\pi\to\mathrm{Isom}(X_k)$ be a sequence of isometric actions of a discrete group π on complete Riemannian n-manifolds X_k such that $\rho_k(\pi)$ acts freely. Suppose that the sequence $(X_k,p_k,\rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X,Γ,p) and (X_k,p_k,ρ_k) converges to (X,p,ρ) in the pointwise convergence topology.*

Then, for any $\epsilon>0$ and for any finite subset $S\subset\pi$, there is $k(\epsilon,S)$ with the property that for each $k>k(\epsilon,S)$ there exists an ϵ–Lipschitz approximation (f_k,g_k,ϕ_k,τ_k) between $(X_k,p_k,\rho_k(\pi))$ and (X,p,Γ) such that $\phi_k(\rho_k(\gamma))=\rho(\gamma)$ and $\rho_k(\gamma)=\tau_k(\rho(\gamma))$ for every $\gamma\in S$.

2.3 Applications of the Margulis' lemma

The following proposition generalizes a well-known statement in the Kleinian group theory. Namely, if π is a non-virtually-abelian group and ρ_k is a sequence of injective discrete representations of π into $\mathbf{PSL}(2,\mathbb{C})$ that converges algebraically to a representation ρ, then ρ is injective and discrete. Moreover, the closure of $\{\rho_k(\pi)\}$ in the Chabauty topology consists of discrete groups.

Proposition 2.5. *Let X_k be a sequence of Hadamard manifolds with sectional curvatures in $[a,b]$ for $a\le b<0$ and let π be a finitely generated group that is not virtually nilpotent. Let $\rho_k:\pi\to\mathrm{Isom}(X_k)$ be an arbitrary sequence of free and isometric actions such that (X_k,p_k,ρ_k) converges converges to (X,p,ρ) in the pointwise convergence topology. Then*

(i) the sequence $(X_k,p_k,\rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology, and

(ii) ρ is a free action, in particular ρ is injective,

Proof. Choose r so that the open ball $B(p,r)\subset X$ contains $\{\rho(\gamma_1)(p),\dots\rho(\gamma_m)(p)\}$ where $\{\gamma_1,\dots\gamma_m\}$ generate π. Passing to subsequence, we assume that $B(p_k,r)$ contains $\{\rho_k(\gamma_1)(p),\dots\rho_k(\gamma_m)(p)\}$.

Show that, for every k, there exists $q_k \in B(p_k, r)$ such that for any $\gamma \in \pi \backslash \{\mathrm{id}\}$, we have $\rho_k(\gamma)(q_k) \notin B(q_k, \mu_n/2)$ where μ_n is the Margulis constant. Suppose not. Then for some k, the whole ball $B(p_k, r)$ projects into the thin part $\{\mathrm{InjRad} < \mu_n/2\}$ under the projection $\pi_k : X_k \to X_k/\rho_k(\pi)$. Thus the ball $B(p_k, r)$ lies in a connected component W of the π_k–preimage of the thin part of $X_k/\rho_k(\pi)$. According to [BGS85, p111], the stabilizer of W in $\rho_k(\pi)$ is virtually nilpotent and, moreover, the stabilizer contains every element $\gamma \in \rho_k(\pi)$ with $\gamma(W) \cap W \neq \emptyset$. Therefore, the whole group $\rho_k(\pi)$ stabilizes W. Hence $\rho_k(\pi)$ must be virtually nilpotent. As ρ_k is injective, π is virtually nilpotent. A contradiction.

Thus, $(X_k, q_k, \rho_k(\pi))$ is Lipschitz precompact [Fuk86] and, hence passing to subsequence, one can assume that $(X_k, q_k, \rho_k(\pi))$ converges to some (X, q, Γ).

It is a general fact that follows easily from definitions that whenever (X_k, q_k, Γ_k) converges to (X, q, Γ) in the equivariant pointed Lipschitz topology and a sequence of points $p_k \in X_k$ converges to $p \in X$, then (X_k, p_k, Γ_k) converges to (X, p, Γ) in the equivariant pointed Lipschitz topology. The proof of (i) is complete.

Pass to a subsequence so that (X_k, p_k, ρ_k) converges to (X, p, Γ) in the equivariant pointed Lipschitz topology. By 2.3, ρ is injective. Furthermore, $\rho(\pi)$ acts freely because it is a subgroup of Γ. Thus, (ii) is proved. □

2.4 Diverging actions and splittings over virtually nilpotent groups

We say that a group π splits over a virtually nilpotent group if π has a nontrivial decomposition into an amalgamated product or an HNN-extension over a virtually nilpotent group.

Let π be a finitely generated group and let $S \subset \pi$ be a finite subset that contains $\{\mathrm{id}\}$ and generates π. Let $\rho_k : \pi \to \mathrm{Isom}(X_k)$ be an arbitrary sequence of free and isometric actions of π on Hadamard n-manifolds X_k. Assume that the sectional curvatures of X_k lie in $[a, b]$ for $a \le b < 0$.

For $x \in X_k$, we denote $D_k(x)$ the diameter of the set $\rho_k(S)(x)$. Set $D_k = \inf_{x \in X_k} D_k(x)$. Suppose the sequence D_k is bounded. Then there exist $x_k \in X_k$ such that $D_k(x_k)$ is bounded. Therefore, as we observed in the section 2.2, the sequence (X_k, x_k, ρ_k) is precompact in the pointwise convergence topology. The following lemma shows what happens if D_k is unbounded.

Proposition 2.6. *Suppose that the sequence $\{D_k\}$ is unbounded. Assume that π is not virtually nilpotent. Then π acts on a certain $\mathbb{R}$-tree without global fixed*

points and so that the stabilizer of any non-degenerate arc is virtually nilpotent. Furthermore, if π is finitely presented, then π splits over a virtually nilpotent group.

Proof. This proposition is well-known to experts. First, using work of Bestvina [Bes88] and Paulin [Pau88, Pau91], we produce an action of π on a real tree and then invoke Rips' machine to get a splitting over a virtually nilpotent group. For completeness, we briefly review the argument.

The rescaled pointed Hadamard manifold $\frac{1}{D_k} \cdot X_k$ has sectional curvature $\leq b \cdot D_k \to -\infty$ as $k \to \infty$. Find $p_k \in X_k$ such that $D_k(p_k) \leq D_k + 1/k$. Consider the sequence of triples $(\frac{1}{D_k} X_k, p_k, \rho_k)$. Repeating an argument of Paulin [Pau91, §4], we can pass to subsequence that converges to a triple $(X_\infty, p_\infty, \rho_\infty)$. (For the definition of the convergence see [Pau88, Pau91]. Paulin calls it "convergence in the Gromov topology".)

The limit space X_∞ is a length space of curvature $-\infty$, that is a real tree. Because of the way we rescaled, the limit space has a natural isometric action ρ_∞ of π with no global fixed point [Pau88, Pau91]. Then it is a standard fact that there exists a unique π–invariant subtree T of X_∞ that has no proper π–invariant subtree. In fact T is the union of all the axes of all hyperbolic elements in π. Since the sectional curvatures are uniformly bounded away from zero and $-\infty$, the Margulis lemma implies that the stabilizer of any non-degenerate segment is virtually nilpotent (cf. [Pau88]).

Note that any increasing sequence of virtually nilpotent subgroups of π is stationary. Indeed, since a virtually nilpotent group is amenable, the union U of an increasing sequence $U_1 \subset U_2 \subset U_3 \subset \dots$ of virtually nilpotent subgroups is also an amenable group. If the fundamental group of a complete manifold of pinched negative curvature is amenable, it must be finitely generated [BS87, Bow93]. In particular, U is finitely generated, hence $U_n = U$ for some n. Thus, the π–action on the tree T is stable [BF95, Proposition 3.2(2)].

We summarize that the π–action on T is stable, has virtually nilpotent arc stabilizers and no proper π–invariant subtree. Since π is finitely presented and not virtually nilpotent, the Rips' machine [BF95, Theorem 9.5] produces a splitting of π over a virtually solvable group. Any amenable subgroup of π must be virtually nilpotent [BS87, Bow93], hence π splits over a virtually nilpotent group. □

Example 2.7. Let K be a closed aspherical n–manifold such that any nilpotent subgroup of $\pi_1(K)$ has cohomological dimension $\leq n-2$. Then $\pi_1(K)$ does not split as a nontrivial amalgamated product or HNN-extension over a

virtually nilpotent group [Bel98]. In particular, this is the case if K is a closed aspherical manifold such that $\dim(K) \geq 3$ and $\pi_1(K)$ is word-hyperbolic.

Theorem 2.8. *Let π be a finitely presented group that is not virtually nilpotent and that does not split over virtually nilpotent subgroups. Suppose that π is not isomorphic to a discrete subgroup of the isometry group of the hyperbolic n-space. Then the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ is empty, for some $\epsilon > 0$.*

Proof. Arguing by contradiction, we assume that for each k there exists a manifold $X_k/\rho_k(\pi) \in \mathcal{M}_{-1-1/k,-1,\pi,n}$ where ρ_k is free, isometric action π on the Hadamard manifold X_k.

According to 2.6, (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology, for some $p_k \in X_k$. Pass to subsequence so that (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology. By 2.3, ρ is a free action of a discrete group π, in particular $X/\rho(\pi)$ is a manifold with fundamental group isomorphic to π.

By a standard argument X is an inner metric space of curvature ≥ -1 and ≤ -1 in the sense of Alexandrov. Hence [Ale57] implies that X is isometric to the real hyperbolic space. Thus, π is the fundamental group of a hyperbolic manifold $X/\rho(\pi)$ which is a contradiction. $\square$

2.5 Accessibility over virtually nilpotent groups

Delzant and Potyagailo have recently proved a powerful accessibility result which we state below (in torsion-free case) for reader's convenience. The definition 2.9 and the theorem 2.10 are taken from [DP98]. I am grateful to Thomas Delzant for helpful discussions.

Recall that a graph of groups is a graph whose vertices and edges are labeled with *vertex groups* π_v and *edge groups* π_e and such that every pair (v, e) where the edge e is incident to the vertex v is labeled with a group monomorphism $\pi_e \to \pi_v$. We only consider finite connected graphs of groups. To each graph of groups one can associate its fundamental group which is a result of repeated amalgamated products and HNN–extensions of vertex groups over the edge groups (see [Bau93] for more details).

Definition 2.9. *A class $\mathcal{E}$ of subgroups of a torsion free group π is called elementary provided the following four conditions hold.*

(i) *$\mathcal{E}$ is closed under conjugation in π;*

(ii) *any infinite group from $\mathcal{E}$ is contained in a* **unique** *maximal subgroup from the class $\mathcal{E}$;*

(iii) *if a group from the class $\mathcal{E}$ acts on a tree, it fixes a point, an end, or a pair of ends;*

(iv) *each maximal infinite subgroup from $\mathcal{E}$ is equal to its normalizer in π.*

Note that the condition (iii) holds if any group in $\mathcal{E}$ is amenable [Neb88].

Theorem 2.10. *Let π be a torsion-free finitely presented group and $\mathcal{E}$ be an elementary class of subgroups of π. Then there exists an integer $K > 0$ and a finite sequence $\pi_0, \pi_1, \ldots, \pi_m$ of subgroups of π such that*

(1) *$\pi_m = \pi$, and*

(2) *for each k with $0 \le k < K$, the group π_k either belongs to $\mathcal{E}$ or does not split as a nontrivial amalgamated product or an HNN-extension over a group from $\mathcal{E}$, and*

(3) *for each k with $K \le k \le m$, the group π_k is the fundamental group of a finite graph of groups with edge groups from $\mathcal{E}$, vertex groups from $\{\pi_0, \pi_1, \ldots, \pi_{k-1}\}$, and proper edge-to-vertex homomorphisms.*

Proposition 2.11. *Let π be a finitely presented group. If $\mathcal{M}_{a,b,\pi,n} \neq \emptyset$, then the class of virtually nilpotent subgroup of π is elementary.*

Proof. The proof is straightforward and can be found in [Bel97a]. □

Proposition 2.12. *Let π be the fundamental group of a finite graph of groups with virtually nilpotent edge groups. Assume $\mathcal{M}_{a,b,\pi,n} \neq \emptyset$. Then*

(1) *π is finitely presented iff all the vertex groups are finitely presented, and*

(2) $\dim_{\mathbb{Q}} H^*(\pi, \mathbb{Q}) < \infty$ *iff* $\dim_{\mathbb{Q}} H^*(\pi_v, \mathbb{Q}) < \infty$ *for every vertex group π_v.*

Proof. Since $\mathcal{M}_{a,b,\pi,n} \neq \emptyset$, any virtually nilpotent subgroup of π is finitely generated [Bow93] and hence is the fundamental group of a closed aspherical manifold [FH81]. The statement (2) now follows from the Mayer-Vietoris sequence. For details on the proof of (1) see [Bel97a]. □

3 Invariants of maps and actions

This section is a condensed version of [Bel98, sections 3;4].

3.1 Invariants of continuous maps

Definition 3.1. *Let B be a topological space and S_B be a set. Let ι be a map that, given a smooth manifold N, and a continuous map from B into N, produces an element of S_B. We call ι an invariant of maps of B if the two following conditions hold:*

(1) *Homotopic maps $f_1 : B \to N$ and $f_2\colon B \to N$ have the same invariant.*

(2) *Let $h\colon N \to L$ be a diffeomorphism of N onto an open subset of L. Then, for any continuous map $f\colon B \to N$, the maps $f\colon B \to N$ and $h \circ f\colon B \to L$ have the same invariant.*

There is a version of this definition for maps into oriented manifolds. Namely, we require that the target manifold is oriented and the diffeomorphism h preserves orientation. In that case we say that ι is an *invariant of maps into oriented manifolds.*

Example 3.2. (Tangent bundle.) Assume B is paracompact and S_B is the set of isomorphism classes of real vector bundles over B. Given a continuous map $f\colon B \to N$, set $\tau(f\colon B \to N) = f^{\#}TN$, the isomorphism class of the pullback of the tangent bundle to N under f. Clearly, τ is an invariant.

Example 3.3. (Intersection number in oriented n-manifolds.) Assume B is compact and fix two homology classes $\alpha \in H_m(B)$ and $\beta \in H_{n-m}(B)$. (In this paper we always use singular (co)homology with integer coefficients unless stated otherwise.) Let $f\colon B \to N$ be a continuous map of a compact topological space B into an oriented n-manifold N where $\dim(N) = n$. Set $I_{n,\alpha,\beta}(f)$ to be the intersection number of $f_*\alpha$ and $f_*\beta$ in N. It was verified in [Bel98] that $I_{n,\alpha,\beta}$ is an integer-valued invariant of maps into oriented manifolds.

We say that an invariant of maps is *liftable* if in the part (2) of the definition the word "diffeomorphism" can be replaced by a "covering map". For example, tangent bundle is a liftable invariant. Intersection numbers are not liftable. The following proposition shows to what extent it can be repaired.

Proposition 3.4. [Bel98] *Let $p : \tilde{N} \to N$ be a covering map of manifolds and let B be a finite connected CW-complex. Suppose that $f : B \to \tilde{N}$ is a map such that $p \circ f : B \to N$ is an embedding (i.e., a homeomorphism onto its image). Then $\iota(f) = \iota(p \circ f)$ for any invariant of maps ι.*

3.2 Invariants of actions

Assume X is a smooth contractible manifold and let $\mathrm{Diffeo}(X)$ be the group of all self-diffeomorphisms of X equipped with the compact-open topology. Let π be the fundamental group of a finite-dimensional CW-complex K with the universal cover $\tilde{K}$.

To any action $\rho\colon \pi \to \mathrm{Diffeo}(X)$, we associate a continuous ρ-equivariant map $\tilde{K} \to X$ as follows. Consider the X–bundle $\tilde{K} \times_\rho X$ over K where $\tilde{K} \times_\rho X$ is the quotient of $\tilde{K} \times X$ by the following action of π

$$\gamma(\tilde{k}, x) = (\gamma(\tilde{k}), \rho(\gamma)(x)), \quad \gamma \in \pi.$$

Since X is contractible, the bundle has a section that is unique up to homotopy through sections. Any section can be lifted to a ρ-equivariant continuous map $\tilde{K} \to \tilde{K} \times X$. Projecting to X, we get a ρ-equivariant continuous map $\tilde{K} \to X$. Note that any two ρ-equivariant continuous maps $\tilde{g}, \tilde{f}\colon \tilde{K} \to X$, are ρ-equivariantly homotopic. (Indeed, $\tilde{f}$ and $\tilde{g}$ descend to sections $K \to \tilde{K} \times_\rho X$ that must be homotopic. This homotopy lifts to a ρ–equivariant homotopy of $\tilde{f}$ and $\tilde{g}$.)

Assume now that $\rho(\pi)$ acts freely and properly discontinuously on X. Then the map $\tilde{f}$ descends to a continuous map $f\colon K \to X/\rho(\pi_1(K))$. We say that ρ is *induced* by f.

Let ι be an invariant of continuous maps of K. Given an action ρ such that $\rho(\pi)$ acts freely and properly discontinuously on X, set $\iota(\rho)$ to be $\iota(f)$ where ρ is induced by f. We say ρ is an *invariant of free, proper discontinuous actions of* $\pi_1(K)$. Similarly, any invariant ι of continuous maps of K into oriented manifolds defines an invariant of free, proper discontinuous, orientation-preserving actions on X.

Note that actions conjugate by a diffeomorphism ϕ of X have same invariants. (Indeed, if $\tilde{f}\colon \tilde{K} \to X$ is a ρ-equivariant map, the map $\phi \circ \tilde{f}$ is $\phi \circ \rho \circ \phi^{-1}$-equivariant.) The same is true for invariants of orientation-preserving actions when ϕ is orientation-preserving.

Example 3.5. (Tangent bundle.) Let τ be the invariant of maps defined in 3.2. Then, for any action ρ such that $\rho(\pi)$ acts freely and properly discontinuously on X, let $\tau(\rho)$ be the pullback of the tangent bundle to $X/\rho(\pi)$ via a map $f\colon K \to X/\rho(\pi)$ that induces ρ.

Example 3.6. (Intersection number for orientation preserving actions.) Assume the cell complex K is finite and choose an orientation on X (which makes sense because, like any contractible manifold, X is orientable). Given

homology classes $\alpha \in H_m(K)$ and $\beta \in H_{n-m}(K)$, let $I_{n,\alpha,\beta}$ is an invariant of maps defined in 3.3 where $n = \dim(X)$.

Let ρ be an action of π into the group of orientation preserving diffeomorphisms of X such that $\rho(\pi)$ acts freely and properly discontinuously on X. Then let $I_{n,\alpha,\beta}(\rho)$ be the intersection number of $f_*\alpha$ and $f_*\beta$ in $X/\rho(\pi)$ where $f\colon K \to X/\rho(\pi)$ is a map that induces ρ.

3.3 Main theorem

Throughout this section K is a finite, connected CW-complex with a reference point q. Let $\tilde{K}$ be the universal cover of K, $\tilde{q} \in \tilde{K}$ be a preimage of $q \in K$. Using the point $\tilde{q}$ we identify $\pi_1(K,q)$ with the group of automorphisms of the covering $\tilde{K} \to K$. Let ι is an invariant of actions.

Theorem 3.7. *Let $\rho_k : \pi = \pi_1(K,q) \to \mathrm{Isom}(X_k)$ be a sequence of isometric actions of $\pi_1(K)$ on Hadamard n-manifolds X_k such that $\rho_k(\pi)$ is a discrete subgroup of $\mathrm{Isom}(X_k)$ that acts freely. Suppose that, for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X,p,Γ) and (X_k, p_k, ρ_k) converges to (X,p,ρ) in the pointwise convergence topology. Let $\tilde{h} : \tilde{K} \to X$ be a ρ-equivariant continuous map and let $\bar{h} : K \to X/\Gamma$ be the drop of $\tilde{h}$. Then $\iota(\rho_k) = \iota(\bar{h})$ for all large k.*

Proof. Recall that according to 3.2 such a map $\tilde{h}$ always exists and is unique up to ρ-equivariant homotopy.

Let $\tilde{q} \in F \subset \tilde{K}$ be a finite subcomplex that projects onto K. Clearly, the finite set $S = \{\gamma \in \pi_1(K,q) : \gamma(F) \cap F \neq \emptyset\}$ generates $\pi_1(K,q)$.

Note that X is contractible. (Indeed, any spheroid in X lies in the diffeomorphic image of a metric ball in X_j. Any metric ball in a Hadamard manifold is contractible. Thus $\pi_*(X) = 1$.)

Choose $\epsilon > 0$ so small that $\tilde{h}(F)$ lies in the open ball $B(p, 1/10\epsilon) \subset X$. For large k, we find an ϵ-Lipschitz approximation $(\tilde{f}_k, \tilde{g}_k, \phi_k, \tau_k)$ between $(X_k, p_k, \rho_k(\pi))$ and (X, p, Γ).

By lemma 2.4 we can assume that $\tau_k(\rho(\gamma)) = \rho_k(\gamma)$ for all $\gamma \in S$. Hence, the map $\tilde{h}_k = \tilde{g}_k \circ \tilde{h} : F \to X_k$ is ρ_k-equivariant. Extend it by equivariance to a ρ_k-equivariant map $\tilde{h}_k : \tilde{K} \to X_k$. Passing to quotients we get a map $h_k : K \to X_k/\rho_k(\pi_1(K))$ such that $\iota(\rho_k) = \iota(h_k)$.

By construction, $h_k = g_k \circ \bar{h}$ where g_k is the drop of $\tilde{g}_k$. Since g_k is a diffeomorphism, $\iota(h_k) = \iota(\bar{h})$. Hence $\iota(\rho_k) = \iota(\bar{h})$ and the proof is complete. □

Remark 3.8. There is a version of the theorem 3.7 for invariants of maps into oriented manifolds. Suppose all the actions ρ_k on X_k preserve orientations (it makes sense because being a contractible manifold X_k is orientable). Fix an orientation on X (which is also contractible) and choose orientations of X_k so that diffeomorphisms g_k preserve orientations. Then the same proof gives $\iota(\rho_k) = \iota(\bar{h})$ for any invariant of maps into oriented manifolds ι. Yet this new orientation on X_k may be different from the original one.

Corollary 3.9. *Suppose that in addition to the assumptions of the theorem* 3.7 *any of the following holds*

- ι *is a liftable invariant, or*
- $\rho(\pi_1(K)) = \Gamma$, *or*
- $\bar{h}$ *is homotopic to an embedding.*

Then $\iota(\rho_k) = \iota(\rho)$ *for all large* k.

Proof. Let $h : K \to X/\rho(\pi)$ be the drop of $\tilde{h}$. Since $\iota(\rho) = \iota(h)$, it suffices to understand when $\iota(h) = \iota(\bar{h})$. This is trivially true if ι is liftable or if $\rho(\pi_1(K)) = \Gamma$. In case $\bar{h}$ is homotopic to an embedding, 3.4 implies that $\iota(h) = \iota(\bar{h})$. ☐

Remark 3.10. In Kleinian group theory the condition $\rho(\pi_1(K)) = \Gamma$ means, by definition, that ρ_k converges to ρ *strongly.*

4 Applications

4.1 Strongly pinched manifolds have zero Pontrjagin classes

Let X be the limit of Hadamard n-manifolds X_k in the pointed Lipschitz topology. Assume that the sectional curvature of X_k is within $[-1-1/k, -1]$. Then X is a smooth n-manifold [Fuk86] and by a standard argument X is an inner metric space of curvature ≥ -1 and ≤ -1 in the sense of Alexandrov. Hence [Ale57] implies that X is isometric to the real hyperbolic space.

Theorem 4.1. *Let* π *be the fundamental group of a finite aspherical complex. Suppose that* π *is not virtually nilpotent and that* π *does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent group.*

Then, for any positive integer n, there exists an $\epsilon = \epsilon(\pi, n) > 0$ such that any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ is tangentially homotopy equivalent to a manifold of constant negative curvature.

Proof. Arguing by contradiction, consider a sequence of manifolds $X_k/\rho_k(\pi) \in \mathcal{M}_{-1-1/k,-1,\pi,n}$ that are not tangentially homotopy equivalent to manifolds of constant negative curvature.

According to 2.6, (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology, for some $p_k \in X_k$. Hence 2.5 implies that $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology. Pass to subsequence so that (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology and $(X_k, p_k, \rho_k(\pi))$ converges to (X, p, Γ) in the equivariant pointed Lipschitz topology.

By 2.3, ρ is a free action of a discrete group π, in particular $X/\rho(\pi)$ is a manifold with fundamental group isomorphic to π. Set $K = X/\rho(\pi)$ and let τ be the invariant of representations of $\pi_1(K)$ defined in 3.5. According to 3.9, $\tau(\rho) = \tau(\rho_k)$ for large k, thus $\rho_k \circ \rho^{-1}$ induces a tangential homotopy equivalence of $X_k/\rho_k(\pi)$ and $X/\rho(\pi)$. Since X is isometric to the real hyperbolic space, the proof is complete. □

Theorem 4.2. *Let π be the fundamental group of a finite-dimensional aspherical CW-complex K such that $\dim_{\mathbb{Q}} \oplus_m H^{4m}(K, \mathbb{Q}) < \infty$. Suppose that π is finitely presented, not virtually nilpotent and that π does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent group. Then, for any positive integer n, there exists an $\epsilon = \epsilon(\pi, n) > 0$ such that the tangent bundle of any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ has zero rational Pontrjagin classes.*

Proof. Arguing by contradiction, consider a sequence of manifolds $X_k/\rho_k(\pi) \in \mathcal{M}_{-1-1/k,-1,\pi,n}$ whose tangent bundles have nonzero rational Pontrjagin classes. According to 2.6, (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology, for some $p_k \in X_k$. Hence 2.5 implies that $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology. Pass to subsequence so that (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology and $(X_k, p_k, \rho_k(\pi))$ converges to (X, p, Γ) in the equivariant pointed Lipschitz topology.

By an elementary homological argument there exists a finite connected subcomplex $X \subset K$ such that the inclusion $i : X \to K$ induces a π_1–epimorphism and a $\oplus_m H^{4m}(-, \mathbb{Q})$-monomorphism (for example, see [Bel97a] for a proof). The

sequence of isometric actions $\rho_k \circ i_*$ of $\pi_1(X, x)$ on X_k satisfies the assumptions of 3.7, therefore, $\tau(\rho_k \circ i_*) = \tau(\rho \circ i_*)$ for all large k. In other words, the pullback bundles $i^\# \tau(\rho_k)$ are isomorphic to the vector bundle $i^\# \tau(\rho)$.

By 2.3, ρ is a free action of a discrete group π, in particular $X/\rho(\pi)$ is a real hyperbolic manifold. Hence, the tangent bundle to $X/\rho(\pi)$ has zero rational Pontrjagin classes p_m for $m > 0$. (Avez observed in [Ave70] that Pontrjagin forms on any conformally flat manifold vanish.) Hence $\tau(\rho)$ and $i^\# \tau(\rho)$ have zero rational Pontrjagin classes because they are pullbacks of the tangent bundle to $X/\rho(\pi)$. Thus $i^\# \tau(\rho_k)$ has zero rational Pontrjagin classes for all large k. Hence $i^* p_m(\tau(\rho_k)) = p_m(i^\# \tau(\rho_k)) = 0$ all large k. Since i^* is injective, $p_m(\tau(\rho_k)) = 0$ which is a contradiction. ☐

Remark 4.3. For any connected finite-dimensional cell complex B, the Pontrjagin character defines an isomorphism $\widetilde{KO}(B) \to \oplus_{m>0} H^{4m}(B, \mathbb{Q})$. In particular, a vector bundle ξ has zero rational Pontrjagin classes iff for some n the Whitney sum $\underbrace{\xi \oplus \xi \oplus \cdots \oplus \xi}_{n}$ is a trivial bundle.

Proposition 4.4. *Let π be the fundamental group of a finite graph of groups such that each edge group has cohomological dimension ≤ 2. Let n be a positive integer. Assume that there exists an $\epsilon > 0$ such that, for any vertex group π_v, the tangent bundle of any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi_v,n}$ has zero rational Pontrjagin classes. Then the tangent bundle of any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ has zero rational Pontrjagin classes.*

Proof. Following [SW79], we assemble the cell complexes $K(\pi_v, 1)$ and $K(\pi_e, 1) \times [-1, 1]$ into an $K(\pi, 1)$ cell complex by using edge–to–vertex monomorphisms. By Mayer-Vietoris sequence the map $H^{4m}(\pi, \mathbb{Q}) \to \oplus_v H^{4m}(\pi_v, \mathbb{Q})$ induced by inclusions $\pi_v \to \pi$ is an isomorphism. Let $N \in \mathcal{M}_{-1-\epsilon,-1,\pi,n}$ and let N_v be the cover of N that corresponds the inclusions $\pi_v \to \pi$; clearly $N_v \in \mathcal{M}_{-1-\epsilon,-1,\pi_v,n}$. Since coverings preserve tangent bundles, the map $H^{4m}(N, \mathbb{Q}) \to H^{4m}(N_v, \mathbb{Q})$ takes $p_m(TN)$ to $p_m(TN_v)$. Thus, $p_m(TN) = 0$ iff $p_m(TN_v) = 0$ for all v. ☐

Theorem 4.5. *Let π be a finitely presented group with finite $4k$th Betti numbers for all k Assume that any nilpotent subgroup of π has cohomological dimension ≤ 2. Then for any n there exist $\epsilon = \epsilon(n, \pi) > 0$ such that the tangent bundle of any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ has zero rational Pontrjagin classes.*

Proof. Applying the theorem 2.10, we get a sequence $\pi_0, \pi_1, \dots, \pi_m$ of subgroups of π. In particular, for every k with $0 \le k < K$, the group π_k either is virtually nilpotent or does not split over a virtually nilpotent subgroup of π. By the proposition 2.12, the group π_k is finitely presented and has finite Betti numbers. Therefore, by 4.2, any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi_k,n}$ has zero rational Pontrjagin classes.

Repeatedly applying 4.4, we deduce that any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ has zero rational Pontrjagin classes. □

Corollary 4.6. *Let π be a word-hyperbolic group. Then for any n there exist $\epsilon = \epsilon(n,\pi) > 0$ such that the tangent bundle of any manifold in the class $\mathcal{M}_{-1-\epsilon,-1,\pi,n}$ has zero rational Pontrjagin classes.*

Proof. Any torsion free word-hyperbolic group is the fundamental group of a finite aspherical cell complex [CDP90, 5.24]. Moreover, any virtually nilpotent subgroup of a torsion free word-hyperbolic group of is either trivial or infinite cyclic. So 4.5 applies. □

4.2 Intersections in negatively curved manifolds

Given an oriented n-manifold N, consider the intersection form $I_{m,n-m}(\cdot,\cdot)$ of type $(m, n-m)$

$$I_{m,n-m} : H_m(N) \otimes H_{n-m}(N) \to \mathbb{Z}.$$

A homotopy equivalence f of orientable n-manifolds is called $(m, n-m)$-*intersection preserving* if, for some choice of orientations of the manifolds,

$$I_{m,n-m}(f_*\alpha, f_*\beta) = I_{m,n-m}(\alpha,\beta)$$

for all $(\alpha,\beta) \in H_m(N) \otimes H_{n-m}(N)$.

We now extend the notion of $(m, n-m)$-intersection preserving homotopy equivalence to non-orientable manifolds. Recall that a homotopy equivalence of manifolds $f : N \to L$ is called *orientation-true* if the vector bundles TN and $f^{\#}TL$ have equal first Stiefel-Whitney class. Let f be an orientation-true homotopy equivalence of non-orientable manifolds. Then f lifts to a homotopy equivalence $\tilde{f} : \tilde{N} \to \tilde{L}$ of orientable two-fold covers. We say that f is $(m, n-m)$*-intersection preserving* if so is $\tilde{f}$.

If f is $(m, n-m)$*-intersection preserving* for all m, we say that f is *intersection preserving*. For example, if f is homotopic to a homeomorphism, then f is $(m, n-m)$-intersection preserving for all m [Dold, 13.21].

Theorem 4.7. *Let K be a connected finite cell complex and let $I_{n,\alpha,\beta}$ be the invariant of maps defined in* 3.3. *Let $\rho_k : \pi_1(K) \to \mathrm{Isom}(X_k)$ be a sequence of free, isometric actions of $\pi_1(K)$ on Hadamard n-manifolds X_k. Suppose that, for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi_1(K)))$ is precompact in the equivariant pointed Lipschitz topology and (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology.*

Then, for any sequence of continuous maps $f_k : K \to X_k/\rho_k(\pi_1(K))$ that induce ρ_k, the sequence of integers $I_{n,\alpha,\beta}(f_k)$ is bounded.

Proof. Since $I_{n,\alpha,\beta}(f_k) = I_{n,\alpha,\beta}(\rho_k)$, this is a particular case of 3.7. □

Theorem 4.8. *Let π finitely generated group and let $\rho_k : \pi \to \mathrm{Isom}(X_k)$ be a sequence of free, isometric actions of π on Hadamard n-manifolds X_k. Suppose that, for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology and (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology. Assume that the Betti numbers b_m and b_{n-m} of π are finite. Then the set of manifolds $\{X_k/\rho_k(\pi)\}$ falls into finitely many $(m, n-m)$-intersection preserving homotopy types.*

Proof. Argue by contradiction. We can assume that no two manifold from $\{X_k/\rho_k(\pi)\}$ are $(m, n-m)$-intersection preserving homotopy equivalent. Set $K = X_1/\rho_1(\pi)$. Let $f_k : K \to X_k/\rho_k(\pi)$ be a homotopy equivalence that induces ρ_k. We next show that, after passing to a subsequence, the homotopy equivalence $f_{k+1} \circ f_k^{-1}$ is $(m, n-m)$-intersection preserving which yields a contradiction.

Since $\pi_1(K)$ is finitely generated, the group $H^1(K, \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(K), \mathbb{Z}_2)$ is finite. So passing to subsequence we can assume that the vector bundles $f_k^{\#} TX_k/\rho_k(\pi)$ have the same first Stiefel-Whitney class w. Hence the homotopy equivalences $f_{k+1} \circ f_k^{-1}$ are orientation-true.

The first Stiefel-Whitney class w defines a two-fold-cover $\tilde{K} \to K$ and an index two subgroup $\tilde{\pi} = \pi_1(\tilde{K})$ of $\pi_1(K)$. Restricting ρ_k to $\tilde{\pi}$ we get a sequence of free, isometric actions of $\tilde{\pi}$ on Hadamard n-manifolds X_k. Notice that $(X_k, p_k, \rho_k(\tilde{\pi}))$ is precompact in the equivariant pointed Lipschitz topology and $(X_k, p_k, \rho_k|_{\tilde{\pi}})$ is precompact in the pointwise convergence topology. Let $\tilde{f}_k : \tilde{K} \to X_k/\rho_k(\tilde{\pi})$ be a homotopy equivalence that induces $\rho_k|_{\tilde{\pi}}$.

Take arbitrary $\alpha \in H_m(\tilde{K})$ and $\beta \in H_{n-m}(\tilde{K})$ and let I_k be the intersection number of $\tilde{f}_{k*}\alpha$ and $\tilde{f}_{k*}\beta$ in $X_k/\rho_k(\tilde{\pi})$. By an elementary homological argument there exists a finite connected subcomplex K' of $\tilde{K}$ such that the

inclusion $i : K' \subset K$ induces epimorphisms of the fundamental groups and $i_* H_*(K')$ contains α and β (for example, see [Bel97a] for a proof).

Let $i_*\alpha' = \alpha$ and $i_*\beta' = \beta$. Clearly, I_k is equal to the intersection number of $\tilde{f}_{k*} i_* \alpha'$ and $\tilde{f}_{k*} i_* \beta'$ in $X_k/\rho_k(\tilde{\pi})$. According to 4.7, the set of integers $\{I_k\}$ is finite. Hence, passing to subsequence, we can assume that the intersection number of $\tilde{f}_{k*}\alpha$ and $\tilde{f}_{k*}\beta$ in $X_k/\rho_k(\tilde{\pi})$ is independent of k.

The groups $H_m(\tilde{K})$ and $H_{n-m}(\tilde{K})$ have finite rank since the Betti numbers b_m and b_{n-m} of $\tilde{K}$ are finite. (Recall that an abelian group A has finite rank if there exists a finite subset $S \subset A$ such that any non-torsion element of A is a linear combination of elements of S.) Therefore, the intersection form is determined by intersection numbers of finitely many homology classes. (Torsion elements do not matter because the intersection number of a torsion class and any other class is zero.) Then the argument of the previous paragraph implies that, passing to subsequence, we can assume that the intersection number of $\tilde{f}_{k*}\alpha$ and $\tilde{f}_{k*}\beta$ in $X_k/\rho_k(\tilde{\pi})$ is independent of k for any classes α and β. In other words, $\tilde{f}_{k+1} \circ \tilde{f}_k^{-1}$ is an $(m, n-m)$-intersection preserving homotopy equivalence. □

Corollary 4.9. *Let π be a finitely generated group with finite Betti numbers. Assume $\rho_k \colon \pi \to \mathrm{Isom}(X_k)$ is a sequence of free, isometric actions of π on Hadamard n-manifolds X_k. Suppose that, for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology and (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology. Then, for each m, the set of manifolds $\{X_k/\rho_k(\pi)\}$ falls into finitely many $(m, n-m)$-intersection preserving homotopy types.* □

Proposition 4.10. *Let N be an orientable pinched negatively curved manifold with virtually nilpotent fundamental group. Then the intersection number of any two homology classes in N is zero.*

Proof. It suffices to prove that N is homeomorphic to $\mathbb{R} \times Y$ for some space Y. First note that any torsion free, discrete, virtually nilpotent group Γ acting on a Hadamard manifolds X of pinched negative curvature must have either one or two fixed points at infinity [Bow93, 3.3.1]. If Γ has only one fixed point, Γ is parabolic and, hence, it preserves all horospheres at the fixed point. Therefore, if H is such a horosphere, X/Γ is homeomorphic to $\mathbb{R} \times H/\Gamma$. If Γ has two fixed points, Γ preserves a bi-infinite geodesic. Hence X/Γ is the total space of a vector bundle over a circle. Thus X/Γ is homeomorphic to $\mathbb{R} \times Y$ for some space Y unless X/Γ is the Möbius band which is impossible since $N = X/\Gamma$ is orientable. □

Corollary 4.11. *Assume that π is a finitely presented group does not split over a virtually nilpotent group. Let $m \le n$ be integers such that the Betti numbers b_m and b_{n-m} of π are finite. Then, for any $a \le b < 0$, the class $\mathcal{M}_{a,b,\pi,n}$ breaks into finitely many $(m, n-m)$-intersection preserving homotopy types.*

Proof. By 4.10 we can assume that π is not virtually nilpotent. Let N_k be an arbitrary sequence of manifolds from $\mathcal{M}_{a,b,\pi,n}$ represented as $X_k/\rho_k(\pi)$. According to 2.6, (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology, for some $p_k \in X_k$. Hence 2.5 implies that $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology and we are done because of 4.8. $\square$

Theorem 4.12. *Let K be a connected finite cell complex such that the group $\pi_1(K)$ does not split over a virtually nilpotent group. Let $I_{n,\alpha,\beta}$ be the invariant of maps defined in* 3.3 *and let $a \le b < 0$ be real numbers. Then, for any sequence of continuous maps $f_k : K \to N_k$ that induce isomorphisms of fundamental groups of K and $N_k \in \mathcal{M}_{a,b,\pi_1(K),n}$, the sequence of integers $I_{n,\alpha,\beta}(f_k)$ is bounded.*

Proof. By 4.10 we can assume that π is not virtually nilpotent. Arguing by contradiction, consider a sequence of maps $f_k \colon K \to N_k$ that induce π_1-isomorphisms of K and manifolds $N_k \in \mathcal{M}_{a,b,\pi_1(K),n}$ and such that no two integers $I_{n,\alpha,\beta}(f_k)$ are equal.

Each map f_k induces a free, properly discontinuous action ρ_k of $\pi_1(K)$ on the universal cover X_k of N_k such that $I_{n,\alpha,\beta}(f_k) = I_{n,\alpha,\beta}(\rho_k)$. According to 2.6, (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology, for some $p_k \in X_k$. Hence 2.5 implies that $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology and we are done because of 4.7. $\square$

4.3 Vector bundles with negatively curved total spaces

The following is a slight generalization of the theorem 1.4. The proof makes use of the classification theorem proved in the appendix.

Theorem 4.13. *Let M be a closed negatively curved manifold of dimension ≥ 3. Suppose $n > \dim(M)$ is an integer and $a \le b < 0$ are real numbers. Let $f_k : M \to N_k$ be a sequence of smooth embeddings of M into manifolds N_k such that for each k*

- *f_k induces a monomorphism of fundamental groups, and*

- *N_k is a complete Riemannian n-manifold with sectional curvatures within $[a, b]$.*

Then the set of the normal bundles $\nu(f_k)$ of the embeddings falls into finitely many isomorphism classes. In particular, up to diffeomorphism, only finitely many manifolds from the class $\mathcal{M}_{a,b,\pi_1(M),n}$ are total spaces of vector bundles over M.

Proof. Passing to covers corresponding to f_{k*}, we can assume that f_k induce isomorphisms of fundamental groups. Arguing by contradiction, assume that $\nu_k = \nu(f_k)$ are pairwise nonisomorphic.

First, reduce to the case when the bundles ν_k are orientable. Since $H^1(M, \mathbb{Z}_2)$ is finite, we can pass to subsequence and assume that the bundles ν_k have equal first Stiefel-Whitney classes. Pass to two-fold covers corresponding to this Stiefel-Whitney class. Then f_k lift to embeddings with orientable normal bundles. Using A.2, we can pass to subsequence so that these normal bundles are pairwise nonisomorphic. Furthermore, a finite cover of M is still a closed negatively curved manifold of dimension ≥ 3 and any finite cover of N_k is a complete Riemannian manifold with sectional curvatures within $[a, b]$. Thus, it suffices to consider the case of orientable ν_k.

Each map f_k induces a free, properly discontinuous action ρ_k of $\pi_1(K)$ on the universal cover X_k of N_k such that $\tau(f_k) = \tau(\rho_k)$. According to 2.6, (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology, for some $p_k \in X_k$. Hence 2.5 implies that $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology. According to 3.7, the set of vector bundles $\{\tau(f_k)\}$ falls into finitely many isomorphism classes.

In particular, there are only finitely many possibilities for the total Pontrjagin class of $\tau(f_k)$. The normal bundle ν_k of the embedding f_k satisfies $\nu_k \oplus TM \cong \tau(f_k)$. Applying the total Pontrjagin class, we get $p(\nu_k) \cup p(TM) = p(\tau(f_k))$. The total Pontrjagin class of any bundle is a unit, hence we can solve for $p(\nu_k)$. Thus, there are only finitely many possibilities for $p(\nu_k)$.

Thus, according to A.1, we can pass to subsequence so that the (rational) Euler classes of ν_k are all different. Denote the integral Euler class by $e(\nu_k)$.

First, assume that M is orientable. Recall that, by definition, the Euler class $e(\nu_k)$ is the image of the Thom class $\tau(\nu_k) \in H^m(N_k, N_k \backslash f_k(M))$ under the map $f_k^* \colon H^m(N_k, N_k \backslash f_k(M)) \to H^m(M)$. According to [Dol72, VIII.11.18] the Thom class has the property $\tau(\nu_k) \cap [N_k, N_k \backslash f_k(M)] = f_{k*}[M]$ where $[N_k, N_k \backslash f_k(M)]$ is the fundamental class of the pair $(N_k, N_k \backslash f_k(M))$ and $[M]$

is the fundamental class of M. Therefore, for any $\alpha \in H_m(M)$, the intersection number of $f_{k*}\alpha$ and $f_{k*}[M]$ in N_k satisfies

$$I(f_{k*}[M], f_{k*}\alpha) = \langle \tau(\nu_k), f_{k*}\alpha \rangle = \langle f^*\tau(\nu_k), \alpha \rangle = \langle e(\nu_k), \alpha \rangle.$$

Since M is compact, $H_m(M)$ is finitely generated; we fix a finite set of generators. The (rational) Euler classes are all different, hence the homomorphisms $\langle e(\nu(f_k)), - \rangle \in \mathrm{Hom}(H_m(M), \mathbb{Z})$ are all different. Then there exists a generator $\alpha \in H_m(M)$ such that $\{\langle e(\nu(f_k)), \alpha \rangle\}$ is an infinite set of integers. Hence $\{I(f_{k*}[M], f_{k*}\alpha)\}$ is an infinite set of integers. Combining 4.12 and 2.7, we get a contradiction.

Assume now that M is nonorientable. Let $q\colon \tilde{M} \to M$ be the orientable two-fold cover. Any finite cover of aspherical manifolds induces an injection on rational cohomology [Bro82, III.9.5(b)]. Hence $e(q^{\#}\nu(f_k)) = q^* e(\nu(f_k))$ implies that the rational Euler classes of the pullback bundles $q^{\#}\nu(f_k)$ are all different, and there are only finitely many possibilities for the total Pontrjagin classes of $q^{\#}\nu(f_k)$.

Furthermore, the bundle map $q^{\#}\nu(f_k) \to \nu(f_k)$ induces a smooth two-fold cover of the total spaces, thus the total space of $q^{\#}\nu(f_k)$ belongs to $\mathcal{M}_{a,b,\pi_1(\tilde{M}),n}$. and we get a contradiction as in the oriented case. □

Remark 4.14. More generally, the theorem 4.13 is true whenever M is a closed smooth aspherical manifold such that no finite index subgroup of $\pi_1(M)$ splits over a virtually nilpotent group. The proof we gave works *verbatim*. In particular, we can take M to be a smooth manifold that satisfies the conditions of 2.7.

Remark 4.15. In some cases it is easy to decide when there exist *infinitely* many vector bundles of the same rank over a given base. Namely, it suffices to check that certain Betti numbers of the base are nonzero. For example, by a simple K-theoretic argument the set of isomorphism classes of rank m vector bundles over a finite cell complex B is infinite provided $m \geq \dim(B)$ and $\oplus_m H^{4m}(B, \mathbb{Q}) \neq 0$. In fact, any element of $\oplus_m H^{4m}(B, \mathbb{Q})$ is the Pontrjagin character of some vector bundle over B. Furthermore, oriented rank two vector bundles over B are in one-to-one correspondence with $H^2(B, \mathbb{Z})$ via the Euler class.

Note that many arithmetic closed real hyperbolic manifolds have nonzero Betti numbers in all dimensions [MR81]. Any closed complex hyperbolic manifold has nonzero even Betti numbers because the powers of the Kähler form are noncohomologous to zero. Similarly, for each k, closed quaternion hyperbolic manifolds have nonzero $4k$th Betti numbers.

5 Digression: pinching invariants

We now reinterpret our results in terms of certain natural pinching invariants. See insightful discussions in [Gro93, Gro91] for more information.

For a smooth manifold N that admits a metric of pinched negative curvature, define $pinch(N) \in [1,\infty)$ to be the infimum of the numbers a/b such that N is diffeomorphic to a manifold in the class $\mathcal{M}_{a,b,\pi_1(N),\dim(N)}$.

Example 5.1. Replacing "diffeomorphic" by "homeomorphic" leads to a different invariant. For all $n \geq 6$, Farrell, Jones, and Ontaneda constructed sequences of closed negatively curved n-manifolds N_k with $pinch(N_k) \to 1$ such that each N_k is homeomorphic but not diffeomorphic to a hyperbolic manifold [FJO98].

Example 5.2. For any $n \geq 4$, Gromov and Thurston [GT87] constructed a sequence of closed n-manifolds N_k such that $pinch(N_k) > 1$ and $pinch(N_k)$ converges to 1 as $k \to \infty$. These manifolds are not diffeomorphic to closed real hyperbolic manifolds. Furthermore, they constructed a sequence of closed negatively curved n-manifolds N_k such that $pinch(N_k) \to \infty$.

Example 5.3. Let ξ_k be a sequence of pairwise non-isomorphic rank m vector bundles over a closed negatively curved manifold M of dimension ≥ 3. Then according to the theorem 1.4, their total spaces $E(\xi_k)$ have the property that $pinch(N_k)$ converges to infinity.

Example 5.4. Suppose that N is a manifold of pinched negative curvature with word-hyperbolic fundamental group and nontrivial total Pontrjagin class. Then theorem 4.6 implies $pinch(N) > 1$.

Let π be the fundamental group of a manifold of pinched negative curvature. Define $pinch_n(\pi) \in [1,\infty)$ to be the infimum over the numbers a/b such that $\mathcal{M}_{a,b,\pi,n} \neq \emptyset$. Clearly, $pinch(N) \geq pinch_{\dim(N)}(\pi_1(N))$. Notice that

$$pinch_n(\pi) \geq pinch_{n+1}(\pi)$$

because if $N \in \mathcal{M}_{a,b,\pi,n}$, then there is a warped product metric on $N \times \mathbb{R}$ such that $N \times \mathbb{R} \in \mathcal{M}_{a,b,\pi,n+1}$ [FJ87]. Clearly, if Γ is a subgroup of π, then $pinch_n(\Gamma) \leq pinch_n(\pi)$.

Example 5.5. Let π be a finitely generated group that does not act on an $\mathbb{R}$-tree with no global fixed point and virtually nilpotent arc stabilizers. Then

according to the propositions 2.5 and 2.6, there always exists a smooth n-manifold N with complete $C^{1,\alpha}$ Riemannian metric of bounded Alexandrov curvature with pinching equal to $pinch_n(\pi)$. Moreover, by a result of Nikolaev [Nik91], this metric on N can be approximated in (nonpointed) Lipschitz topology by complete Riemannian metrics on N whose pinchings converge to $pinch_n(\pi)$. In particular, $pinch(N) = pinch_n(\pi)$.

Furthermore, if $pinch_n(\pi) = 1$, then by 2.8 the manifold N carries a complete real hyperbolic metric. Thus, for a group π as above, either π is the fundamental group of a real hyperbolic manifold, or else $pinch_n(\pi) > 1$.

Example 5.6. Any closed negatively curved n-manifold N with $pinch_n(\pi_1(N)) = 1$ is diffeomorphic to a real hyperbolic manifold. This is obvious if $\dim(N) = 2$ and follows from 2.8 and 2.7 if $\dim(N) > 2$.

Example 5.7. Let π be a (discrete) group with Kazhdan's property (T). Then π is finitely generated and any action of π on an $\mathbb{R}$-tree has a global fixed point [dlHV89]. Furthermore, π is not the fundamental group of a real hyperbolic manifold [dlHV89]. Thus we conclude that $pinch_n(\pi) > 1$.

Sometimes it is possible to compute or at least estimate $pinch_n(\pi)$. Here we only give two examples that use harmonic maps. See [Gro91] for other results in this direction.

Example 5.8. Let M be a closed Kähler manifold such that $\pi_1(M)$ has cohomological dimension > 2. Then $pinch_n(\pi_1(M)) \geq 4$ for all $n \geq \dim(M)$ [YZ91]. If, in addition, M is complex hyperbolic, then $pinch_n(\pi_1(M)) = 4$.

Example 5.9. Let M be a closed quaternion hyperbolic or Cayley hyperbolic manifold. Then $pinch_n(\pi_1(M)) = 4$. Moreover, if π is a quotient of $\pi_1(M)$, then $pinch_n(\pi) \geq 4$ [MSY93].

A Classifying vector bundles

The purpose of this appendix is to prove that the isomorphism type of any vector bundle is determined, up to finitely many possibilities, by the characteristic classes of the bundle. This fact is apparently well-known to experts, yet there seem to be no published proof. The proof given below is elementary and mainly uses obstruction theory. I am most grateful to Jonathan Rosenberg from whom I learned the orientable case and to Sergei P. Novikov for help in the non-orientable case. Here is the precise statement for orientable vector bundles; the proof can be found in [Bel98].

Theorem A.1. *Let K be a finite CW-complex and m be a positive integer. Then the set of isomorphism classes of oriented real (complex, respectively) rank m vector bundles over K with the same rational Pontrjagin classes and the rational Euler class (rational Chern classes, respectively) is finite.*

First, we review the proof for orientable vector bundles of, say, even rank m. Characteristic classes can be thought of as homotopy classes of maps from the classifying space $BSO(m)$ to Eilenberg-MacLane spaces. For example, Euler class and Pontrjagin classes are given by $e \in H^m(BSO(m), \mathbb{Z}) \cong [BSO(m), K(m, \mathbb{Z})]$ and $p_i \in H^{4i}(BSO(m), \mathbb{Z}) \cong [BSO(m), K(4i, \mathbb{Z})]$. The map $(e, p_1, \dots, p_{m/2-1})$ of $BSO(m)$ to the product of Eilenberg-MacLane spaces is known to induce an isomorphism in rational cohomology. Thus, since the spaces are simply-connected, $(e, p_1, \dots, p_{m/2-1})$ is a rational homotopy equivalence. Now the obstruction theory implies that a map of a finite cell complex K into the product of Eilenberg-MacLane spaces can have only finitely many nonhomotopic liftings to $BSO(m)$. In other words, only finitely many vector bundles over K can have the same characteristic classes.

The above argument fails for nonorientable bundles, due to the fact $BO(m)$ is not simply connected (i.e., the map $c = (p_1, \dots, p_{[m/2]})$ of $BO(m)$ to the product of Eilenberg-MacLane spaces is *not* a rational homotopy equivalence even though it induces an isomorphism on rational cohomology). Yet essentially the same result is true. To make a precise statement we need the following background.

Given a finite CW-complex K, the set of isomorphism classes of rank m vector bundles over a K is in one-to-one correspondence with the set of homotopy classes of maps $[K, BO(m)]$; to a map $f : K \to BO(m)$ there corresponds the pullback $f^\# \gamma_m$ of the universal rank m vector bundle γ_m over $BO(m)$. A vector bundle $f^\# \gamma_m$ is orientable iff f lifts to $BSO(m)$ which is a two-fold-cover of $BO(m)$. In terms of characteristic classes a vector bundle is orientable iff its first Stiefel-Whitney class vanishes.

Let $\xi = f^\# \gamma_m$ and $\eta = g^\# \gamma_m$ be nonorientable vector bundles that have the same first Stiefel-Whitney class $w_1(\xi) = w_1(\eta) = w \in H^1(K, \mathbb{Z}_2) \setminus 0$. The universal coefficient theorem provides a natural isomorphism of $H^1(K, \mathbb{Z}_2)$ and $\mathrm{Hom}(H_1(K), \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(K), \mathbb{Z}_2)$. Thus, to a nonzero element of $H^1(K, \mathbb{Z}_2)$ there corresponds an epimorphism of $\pi_1(K)$ onto $\mathbb{Z}_2$ whose kernel is an index two subgroup of $\pi_1(K)$. This index two subgroup defines a two-fold-cover $\tilde{K} \to K$.

Let $p : \tilde{K} \to K$ be the two-fold-cover that corresponds to the class w. Clearly $p^* w = 0$, hence the pullback bundles $p^\# \xi$ and $p^\# \eta$ are orientable. In other

words, the maps $f \circ p$ and $g \circ p$ of $\tilde{K}$ to $BO(m)$ can be lifted to $BSO(m)$. The lifts $\tilde{f}, \tilde{g} : \tilde{K} \to BSO(m)$ are equivariant with respect the covering actions of $\mathbb{Z}_2$. Clearly, f is homotopic to g iff $\tilde{f}$ is equivariantly homotopic to $\tilde{g}$.

Theorem A.2. *Let ξ be nonorientable vector bundle over a finite CW-complex K and let $p : \tilde{K} \to K$ be the two-fold-cover that corresponds to the first Stiefel-Whitney class of ξ. Then ξ is is determined up to finitely many possibilities by the Euler class and the total Pontrjagin class of $p^{\#}\xi$.*

The proof is based on the equivariant obstruction theory which is reviewed below. Suppose $\tilde{K} \to K$ is a two-fold-cover of a finite CW-complex K. Thus, we get an involution on the set of cells of $\tilde{K}$ and hence an involution ι of the cellular chain complex $C_*(\tilde{K})$. Clearly, ι commutes with the boundary homomorphism.

The "usual" cellular cohomology $H^*(K, \Pi)$ of $\tilde{K}$ with coefficients in a finitely generated abelian group Π is the homology of the complex $\mathrm{Hom}(C_*(\tilde{K}), \Pi)$.

Let $\mathbb{Z}_2$ act (by group automorphisms) on Π; in other words there is $i \in \mathrm{Aut}(\Pi)$ such that i^2 is the identity. A cochain $f \in \mathrm{Hom}(C_*(\tilde{K}), \Pi)$ is called *equivariant* if $f(\iota c) = if(c)$. The *equivariant* cellular cohomology $H^*_e(\tilde{K}, \Pi)$ of $\tilde{K}$ with coefficients in Π is the homology of the subcomplex of equivariant cochains

$$\mathrm{Hom}_{\mathbb{Z}_2}(C_*(\tilde{K}), \Pi) \subset \mathrm{Hom}(C_*(\tilde{K}), \Pi).$$

This inclusion induces a homomorphism $H^*_e(\tilde{K}, \Pi) \to H^*(\tilde{K}, \Pi)$.

Lemma A.3. *The homomorphism $H^*_e(\tilde{K}, \Pi) \to H^*(\tilde{K}, \Pi)$ has finite kernel.*

Proof. Denote the complex $\mathrm{Hom}(C_*(\tilde{K}), \Pi)$ by C and the subcomplex of equivariant cochains $\mathrm{Hom}_{\mathbb{Z}_2}(C_*(\tilde{K}), \Pi)$ by C_+. Let $C_- \subset C$ be the subcomplex of anti-equivariant cochains where a cochain f is called *anti-equivariant* if

$$f(\iota c) = -if(c).$$

First notice that the inclusion $C_+ \to C_+ \oplus C_-$ induces a monomorphism in homology. Indeed, take $f \in C_+$ such that $(f, 0)$ is a boundary, that is $(f, 0) = \partial(g, h) = (\partial g, \partial h)$. Thus, $f = \partial g$.

Second, show that the map $C_+ \oplus C_- \to C$ that takes (f, g) to $f + g$ has finite kernel. Indeed, assume $f + g = 0$. Since f is equivariant and g is anti-equivariant we deduce

$$if(c) = f(\iota c) = -g(\iota c) = ig(c).$$

Hence $f = g$ and, therefore, both f and g have order two. Thus, the kernel of $C_+ \oplus C_- \to C$ lies in the 2-torsion of $C_+ \oplus C_-$. In particular, the kernel lies in

the torsion subgroup of $C \oplus C$ which is finite because C is a finitely generated abelian group. (In fact, if $\tilde{K}$ has k cells, $C_*(\tilde{K})$ is a free abelian group of rank k. Hence C is a the direct sum of k copies of a finitely generated group Π.)

Third, prove that the map $C_+ \oplus C_- \to C$ has finite cokernel. Notice that its image $C_+ + C_-$ contains $2C$. (Indeed $2f$ is the sum of an equivariant cochain f_+ and an anti-equivariant cochain f_- where $f_\pm(c)$ is defined as $f(c) \pm f(\iota c)$.) Thus, it suffices to check that $C/2C$ is finite which is true because C is a finitely generated abelian group.

Finally, two short exact sequences of chain complexes

$$0 \to \ker \to C_+ \oplus C_- \to C_+ + C \to 0 \quad \text{and} \quad 0 \to C_+ + C_- \to C \to \operatorname{coker} \to 0$$

induce long exact sequences in homology with $H(\ker)$ and $H(\operatorname{coker})$ finite. Therefore, both $H(C_+ \oplus C_-) \to H(C_+ + C_-)$ and $H(C_+ + C_-) \to H(C)$ have finite kernel. In particular, the composition

$$H(C_+) \to H(C_+ \oplus C_-) \to H(C_+ + C_-) \to H(C)$$

has finite kernel as desired. □

For free, proper discontinuous actions the usual obstruction theory routinely generalizes to the equivariant case [Dug57].

Let $\tilde{K} \to K$ and $\tilde{Y} \to Y$ be two-fold-covers of CW-complexes and let $\tilde{f}$ and $\tilde{g}$ be continuous maps of $\tilde{K}$ into $\tilde{Y}$ that are equivariant with respect to a unique isomorphism of covering groups. Assume that $\tilde{Y}$ is simply-connected. Then $\tilde{f}$ and $\tilde{g}$ are equivariantly homotopic on the one-skeleton [Dug57, Lemma 9.1]. Assume $\tilde{f}$ and $\tilde{g}$ are equivariantly homotopic on the $(n-1)$-skeleton.

Consider the difference cochain $d^n(\tilde{f}, \tilde{g})$ with coefficients in $\pi_n(\tilde{Y})$ that comes from the usual obstruction theory; in fact, $d^n(\tilde{f}, \tilde{g})$ is a cocycle since $\tilde{f}$ and $\tilde{g}$ are defined on the whole $\tilde{K}$. It was shown in [Dug57, p266] that $d^n(\tilde{f}, \tilde{g})$ is an equivariant cochain and, furthermore, if $d^n(\tilde{f}, \tilde{g})$ is an equivariant coboundary, then $\tilde{f}$ and $\tilde{g}$ are equivariantly homotopic on the n-skeleton.

In our case $Y = BO(m)$ and $\tilde{Y} = BSO(m)$. In particular, $\pi_n(BO(m))$ is a finitely generated abelian group hence the lemma A.3 applies.

Proof of the theorem A.2. We are going to argue by contradiction. Let ξ_i be an infinite sequence of vector bundles given by maps $f_i : K \to BO(m)$. Assume that the bundles ξ_i have equal first Stiefel-Whitney classes and let $p : \tilde{K} \to K$ be the two-fold-cover corresponding to this first Stiefel-Whitney class. Assume also that the bundles $p^\# \xi_i$ have the same Euler class and total Pontrjagin class.

As before, let $\tilde{f}_i : \tilde{K} \to BSO(m)$ be the lift of f_i. Note that all the maps $\tilde{f}_i$ are equivariantly homotopic on the one-skeleton [Dug57, Lemma 9.1]. Using the theorem A.1, we pass to subsequence so that all the bundles $p^{\#}\xi_i$ are isomorphic. In other words, all the maps $\tilde{f}_i$ are (non-equivariantly) homotopic.

Let $n > 1$ be the smallest integer such that infinitely many of $\tilde{f}_i$'s are not equivariantly homotopic on the n-skeleton. Pass to subsequence to assume that $\tilde{f}_i$'s are equivariantly homotopic on the $(n-1)$-skeleton. Thus, the difference cochains $d^n(\tilde{f}_i, \tilde{f}_1)$ are defined.

Since all the maps $\tilde{f}_i$ are (non-equivariantly) homotopic, $d^n(\tilde{f}_i, \tilde{f}_1)$ represents the zero element in the (non-equivariant) cohomology. By lemma A.3, we can pass to subsequence so that the difference cochains $d^n(\tilde{f}_i, \tilde{f}_1)$ represents the same element in the equivariant cohomology.

By the properties of the difference cochain $d^n(\tilde{f}_i, \tilde{f}_j) = d^n(\tilde{f}_i, \tilde{f}_1) - d^n(\tilde{f}_j, \tilde{f}_1)$, hence $d^n(\tilde{f}_i, \tilde{f}_j)$ represents the zero element in the equivariant cohomology. Hence, f_i and f_j are equivariantly homotopic on the n-skeleton. This is a contradiction with the assumption that the sequence $\{f_i\}$ is infinite. □

Remark A.4. Note that, if either m is odd or $m > \dim(K)$, the rational Euler class is zero, and, hence rational Pontrjagin classes determine a vector bundle up to a finite number of possibilities. (We use here that $H^1(K, \mathbb{Z}_2)$ is finite.)

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